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# 5-COLOURING GRAPHS WITH 4 CROSSINGS \*

ROK ERMAN<sup>†</sup>, FRÉDÉRIC HAVET<sup>‡</sup>, BERNARD LIDICKÝ<sup>§</sup>, AND ONDŘEJ PANGRÁC<sup>§</sup>

**Abstract.** We answer in the negative a question of Oporowski and Zhao asking whether every graph with crossing number at most 5 and clique number at most 5 is 5-colourable. However, we show that every graph with crossing number at most 4 and clique number at most 5 is 5-colourable. We also show some colourability results on graphs that can be made planar by removing a few edges. In particular, we show that if a graph with clique number at most 5 has three edges whose removal leaves the graph planar, then it is 5-colourable.

**Key words.** chromatic number; crossing number; clique number; girth

**1. Introduction.** The crossing number of a graph  $G$ , denoted by  $\text{cr}(G)$ , is the minimum number of crossings in any drawing of  $G$  in the plane.

The Four Colour Theorem states that, if a graph has crossing number zero, then it is 4-colourable. A natural question is to ask whether the chromatic number of a graph is bounded in terms of its crossing number. To answer the question, the concept of crossing cover is crucial. A *crossing cover* of a drawing of a graph is a set of vertices  $C$  such that every crossing has an edge incident with a vertex in  $C$ . If  $C$  is a crossing cover, then  $G - C$  is planar, so  $\chi(G) \leq 4 + \chi(G[C]) \leq 4 + |C|$ . Picking one vertex per crossing, we obtain a crossing cover of cardinality at most  $\text{cr}(G)$  so  $\chi(G) \leq 4 + \text{cr}(G)$ .

This upper bound is tight only for  $\text{cr}(G) \leq 1$ . So it is natural to ask for the smallest integer  $f(k)$  such that every graph  $G$  with crossing number at most  $k$  is  $f(k)$ -colourable? An argument similar to the one above shows that  $f(k+1) \leq f(k) + 1$ . Settling a conjecture of Albertson [1], Schaefer [14] showed that  $f(k) = O(k^{1/4})$ . This upper bound is tight up to a constant factor since  $\chi(K_n) = n$  and  $\text{cr}(K_n) \leq \binom{E(K_n)}{2} = \binom{n}{2} \leq \frac{1}{8}n^4$ .

The values of  $f(k)$  are known for a number of small values of  $k$ . The Four Colour Theorem states  $f(0) = 4$  and implies easily that  $f(1) \leq 5$ . Since  $\text{cr}(K_5) = 1$ , we have  $f(1) = 5$ . Oporowski and Zhao [13] showed that  $f(2) = 5$ . Since  $\text{cr}(K_6) = 3$ , we have  $f(3) = 6$ . Further, Albertson et al. [2] showed that  $f(6) = 6$ . Albertson then conjectured that if  $\chi(G) = r$ , then  $\text{cr}(G) \leq \text{cr}(K_r)$ . This conjecture was proved by Barát and Tóth [3] for  $r \leq 16$ .

A graph  $G$  is *r-critical* if  $\chi(G) = r$  and  $\chi(G') < r$  for every proper subgraph  $G'$  of  $G$ . Oporowski and Zhao [13] proved that  $K_6$  is the unique 6-critical graph with crossing number 3.

**THEOREM 1.1** (Oporowski and Zhao [13]). *If  $\text{cr}(G) \leq 3$  and  $\omega(G) \leq 5$ , then  $\chi(G) \leq 5$ .*

Oporowski and Zhao [13] asked whether the conclusion remains true even if  $\text{cr}(G) \in \{4, 5\}$ .

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QUESTION 1.2 (Oporowski and Zhao [13]). *If  $\text{cr}(G) \leq 5$  and  $\omega(G) \leq 5$ , is  $G$  5-colourable?*

We answer in the negative by showing a counterexample. The help of Zdeněk Dvořák was greatly appreciated while obtaining the following theorem.

THEOREM 1.3. *There exists a graph  $G$  such that  $\text{cr}(G) = 5$ ,  $\omega(G) \leq 5$  and  $\chi(G) = 6$ .*

On the other hand, we answer Question 1.2 in the affirmative when  $\text{cr}(G) = 4$ .

THEOREM 1.4. *If  $\text{cr}(G) \leq 4$  and  $\omega(G) \leq 5$ , then  $\chi(G) \leq 5$ .*

A key notion in the proof of Theorem 1.4 is the one of dependent crossings. The *cluster* of a crossing is the set of endvertices of its two edges. Two crossings are *dependent* if their clusters intersect.

Settling a conjecture of Albertson [1], Král' and Stacho [12] showed the following.

THEOREM 1.5 (Král' and Stacho [12]). *If a graph  $G$  has a drawing in the plane in which no two crossings are dependent, then  $\chi(G) \leq 5$ .*

Loosely speaking, this theorem states that, if the crossings are far apart from each other, then the graph is 5-colourable. On the other hand, if all the crossings are very close, that is if all their clusters share a common vertex, then the graph is also 5-colourable. In the same vein, we show that if the crossings are covered by  $2k$  edges, then the graph is  $(4+k)$ -colourable (Theorem 4.1). In particular, if the crossings are covered by three edges, then the graph is 6-colourable. This bound 6 is tight since  $\text{cr}(K_6) = 3$  and thus one can remove three edges from  $K_6$  to make it planar. However, by generalizing Theorem 1.1, we show that  $K_6$  is essentially the unique obstruction for such a graph to be 5-colourable.

THEOREM 1.6. *If  $\omega(G) \leq 5$  and there exists a set  $F$  of at most three edges such that  $G \setminus F$  is planar, then  $\chi(G) \leq 5$ .*

Related open problems are discussed in the final section.

## 2. Preliminaries.

**2.1. Drawings of graphs.** A *drawing*  $\tilde{G}$  (in the plane or the sphere) of a graph  $G = (V, E)$  consists of a bijection  $D$  from  $V \cup E$  into a set  $\tilde{V} \cup \tilde{E}$  such that

- (i)  $\tilde{V}$  is the image of  $V$  and a set of distinct points in the plane;
- (ii) for any edge  $e = uv$ , the element  $D(e) = \tilde{e}$  of  $\tilde{E}$  is the image of a continuous injective mapping  $\phi_e$  from  $[0, 1]$  to the plane which is simple (i.e. does not intersect itself) such that  $\phi_e(0) = D(u)$ ,  $\phi_e(1) = D(v)$  and  $\phi_e([0, 1]) \cap \tilde{V} = \emptyset$ ;
- (iii) every point in the plane is in at most two images of edges unless it is in  $\tilde{V}$ ;
- (iv) for two distinct edges  $e_1$  and  $e_2$  of  $E$ ,  $\tilde{e}_1$  and  $\tilde{e}_2$  intersects in a finite number of points.

We shall often confound the vertex and edge sets of a graph with their image in one of its drawings.

A *crossing* in a drawing of  $G$  is a point in the plane minus  $\tilde{V}$  that belongs to two edges. Formally, it is a point of  $\phi_{e_1}([0, 1]) \cap \phi_{e_2}([0, 1])$  for some edges  $e_1$  and  $e_2$ . A *portion* of an edge  $e$  is a subarc of  $\phi_e[0, 1]$  between two consecutive endpoints or crossings on  $e$ . A portion from  $a$  to  $b$  is called an  $(a, b)$ -*portion*.

A graph is *planar* if it has a drawing without crossings. An easy consequence of Euler's Formula is the following well known proposition.

PROPOSITION 2.1. *If  $G$  is planar, then  $|E(G)| \leq 3|V(G)| - 6$ .*

A drawing of  $G$  is *optimal* if it minimizes the number of crossings. Note that two edges may intersect several times, in either endvertices or crossings. However, thanks to the two following lemmas, we will only consider *nice* drawings, i.e. drawings such that two edges intersect at most once.

LEMMA 2.2. *Every graph with crossing number  $k$  has a nice drawing with at most  $k$  crossings.*

*Proof.* Let  $G$  be a graph with crossing number  $k$ . Consider an optimal drawing of  $G$  that minimizes the number of crossings between edges with a common vertex. Suppose, by contradiction, that two edges  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$  intersect at least twice. Let  $a$  and  $b$  be two points in the intersection of  $e_1$  and  $e_2$ . Without loss of generality we may assume that  $u_1, u_2, v_1$ , and  $v_2$  are in the exterior of the closed curve  $C$  which is the union of the  $(a, b)$ -portion  $P_1$  on  $u_1v_1$  and the  $(a, b)$ -portion  $P_2$  on  $u_2v_2$ . We may also assume that  $P_1$  contains at least as many crossings as  $P_2$ .

Then one can redraw  $u_1v_1$  along the  $(u_1, a)$ -portion of  $e_1$ ,  $P_2$ , and the  $(b, v_1)$ -portion of  $e_1$  slightly in the exterior of  $C$  so that  $e_1$  and  $e_2$  do not cross anymore. Doing so, all the crossings of  $P_1$  including  $a$  and  $b$  (if they were crossings) disappear while a crossing is created per crossings of  $P_2$  distinct from  $a$  and  $b$ . Since one of  $\{a, b\}$  must be a crossing (there are no parallel arcs), we obtain a drawing with one crossing less, a contradiction.  $\square$

Similarly, one can show the following lemma.

LEMMA 2.3. *Every graph with a set  $F$  of edges whose deletion results in a planar graph has a nice drawing in which each crossing contains at least one edge from  $F$ .*

In this paper, we consider only nice drawings. Thus a crossing is uniquely defined by the pair of edges it belongs to. Henceforth, we will often confound a crossing with this set of two edges.

A *face* of a drawing  $\tilde{G}$  is a connected component of the space obtained by deleting  $\tilde{V} \cup \tilde{E}$  from the plane. We let  $F(\tilde{G})$  (or simply  $F$ ) be the set of faces of  $\tilde{G}$ . We say that a vertex  $v$  or a portion of an edge is incident to  $f \in \tilde{F}$  if  $v$  is contained in the closure of  $f$ . The boundary of  $f$ , denoted by  $bd(f)$  consists of the vertices and maximum (with respect to inclusion) portions of edges incident to it. An *embedding* of a graph is the set of boundaries of the faces of some drawing of  $G$  in the plane.

LEMMA 2.4. *Up to a permutation of the vertices, there is only one embedding of  $K_6$  using exactly three crossings. (See Figure 2.1.)*

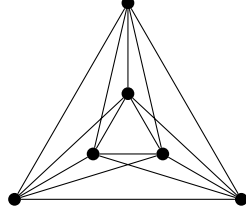
*Proof.* Let  $A$  be an embedding of  $K_6$  using three crossings. Let us show that it is unique. First we observe that every edge is crossed at most once. Otherwise, there will be two edges whose removal leaves the graph planar which is a contradiction to Proposition 2.1. As every cluster of a crossing contains four vertices, there must be a vertex  $v$  contained in two of them. Note that  $v$  cannot be in all three clusters since  $K_6 - v$  (which is isomorphic to  $K_5$ ) is not planar. Let  $e_1 = vx$  and  $e_2 = vy$  be the two crossed edges adjacent to  $v$  and  $e_3$  one of the edges of the crossing whose cluster does not contain  $v$ . The graph  $K_6 \setminus \{e_1, e_2, e_3\}$  is a planar triangulation  $T$  where  $\deg(v) = 3$ .

We denote  $a, b, c$  the neighbours of  $v$  in  $T$ . They must induce a triangle. Without loss of generality,  $ab$  and  $bc$  are the edges crossed by  $e_1$  and  $e_2$ , respectively.

As  $T$  is a triangulation  $abx$  and  $bcy$  form triangles. Moreover,  $xbx$  is also a triangle as  $x$  and  $y$  are consecutive neighbours around  $b$ . The last two edges, which are not discussed yet, are  $xc$  and  $ya$ . They must cross inside  $bxyc$  (one of them is  $e_3$ ). Hence  $A$  is unique.  $\square$

LEMMA 2.5. *A drawing of  $K_5$  with all vertices incident to the same face requires 5 crossings.*

*Proof.* Let us number the vertices of  $K_5$   $v_1, v_2, v_3, v_4, v_5$  in the clockwise order around the boundary of the face  $f$  incident to them. Then free to redraw the edges  $v_1v_2, v_2v_3, v_3v_4, v_4v_5$  and  $v_5v_1$ , we may assume that the boundary is the

FIG. 2.1. Drawing of  $K_6$  with three crossings.

cycle  $v_1v_2v_3v_4v_5$  and that  $f$  is its interior. Now both  $v_1v_3$  and  $v_2v_4$  are in the exterior of  $C$  and thus must cross. Similarly,  $\{v_2v_4, v_3v_5\}$ ,  $\{v_3v_5, v_4v_1\}$ ,  $\{v_4v_1, v_5v_2\}$  and  $\{v_5v_2, v_1v_3\}$  are crossings.  $\square$

LEMMA 2.6. *A drawing of  $K_{2,3}$  such that vertices of each part are in a common face requires at least one crossing.*

*Proof.* Let  $(\{u_1, u_2\}, \{v_1, v_2, v_3\})$  be the bipartition of  $K_{2,3}$ . Suppose by contradiction that  $K_{2,3}$  has a drawing such that each part of the bipartition is in a common face. Then adding a vertex  $u_3$  in the face incident to the vertices  $v_1, v_2$  and  $v_3$  and connecting  $u_3$  to those vertices by new edges yields a drawing of  $K_{3,3}$  with no crossing which contradicts the fact that  $K_{3,3}$  is not planar.  $\square$

**2.2. Properties of 6-critical graphs.** A *stable crossing cover* of a drawing of a graph is a set of vertices that is both stable and a crossing cover.

LEMMA 2.7. *Every graph with a stable crossing cover is 5-colourable.*

*Proof.* Let  $G$  be a graph having a stable crossing cover  $W$ . Use the Four Colour Theorem on  $G - W$  and extend the colouring to  $G$  by using a fifth colour on  $W$ .  $\square$

Let  $G$  be a graph and  $u, v$  be vertices of  $G$ . The operation of *identification* of  $u$  and  $v$  in  $G$  results in a graph denoted by  $G/\{u, v\}$ , which is obtained from  $G - \{u, v\}$  by adding a new vertex  $w$  and the set of edges  $\{wz \mid uz \text{ or } vz \text{ is an edge of } G\}$ .

LEMMA 2.8. *Let  $G$  be a graph and  $v$  be a degree five vertex of  $G$ . Let  $u$  and  $w$  be two non-adjacent neighbours of  $v$ . If  $(G - v)/\{u, w\}$  is 5-colourable, then so is  $G$ .*

*Proof.* A proper 5-colouring of  $(G - v)/\{u, w\}$  corresponds to a proper 5-colouring of  $G - v$  such that  $u$  and  $w$  are coloured by the same colour. So it can be extended to a proper 5-colouring of  $G$  by assigning a colour to  $v$ .  $\square$

Let  $G$  be a graph and a drawing of it in the plane. A cycle is *separating* if it has a vertex in its interior and a vertex in its exterior. A cycle  $C$  is *non-crossed* if all its edges are non-crossed. It is *regular* if any cluster of a crossing containing an edge of  $C$  contains at least three vertices of  $C$ .

LEMMA 2.9. *In every drawing in the plane of a 6-critical graph there is no separating regular triangle.*

*Proof.* Let  $G$  be a 6-critical graph drawn in the plane. Suppose, by way of contradiction, that there is a regular triangle  $C$ . Let  $G_1$  be the graph induced by the vertices in  $C$  and inside  $C$  and let  $G_2$  be a graph induced by the vertices in  $C$  and outside  $C$ . Since  $C$  is separating, both  $G_1$  and  $G_2$  have fewer vertices than  $G$ . Hence, by 6-criticality of  $G$ , they are 5-colourings of those graphs. In addition, in these colourings of  $G_1$  and  $G_2$ , the colours of the vertices of  $C$  are distinct. So, free to permute the colours, one can assume that the two 5-colourings of  $G_1$  and  $G_2$  agree on  $C$ . Hence their union yields a 5-colouring of  $G$ .  $\square$

LEMMA 2.10. *Let  $G$  be a 6-critical graph distinct from  $K_6$ . In every nice drawing of  $G$ , there is no separating triangle such that*

- *at most one of its edges is crossed, and*
- *there is at most one crossing in its interior.*

*Proof.* Suppose, by way of contradiction, that such a cycle  $C = x_1x_2x_3$  exists. Then by Lemma 2.9, one of its edges, say  $x_2x_3$ , is crossed. Let  $uv$  be the edge crossing it with  $u$  inside  $C$  and  $v$  outside. By Lemma 2.9,  $C$  is not regular, so  $u \neq x_1$ . Moreover,  $u \notin \{x_2, x_3\}$  since the drawing is nice.

Let  $G_1$  be the graph induced by  $C$  and the vertices outside  $C$ . Then  $G_1$  admits a 5-colouring  $c_1$  since  $G$  is 6-critical.

Let  $G_2$  be the graph obtained from the graph induced by  $C$  and the vertices inside  $C$  by adding the edges  $ux_1$ ,  $ux_2$  and  $ux_3$  if they do not exist. Observe that  $G_2$  has a planar drawing with at most 2 crossings. Indeed the edge  $ux_1$  may be drawn along  $uv$  and then a path in the outside of  $C$  and the edges  $ux_2$  and  $ux_3$  may be drawn along the edges of the crossing  $\{x_2x_3, uv\}$ . Thus  $G_2$  admits a 5-colouring  $c_2$ .

In both colourings, the colours of the vertices of  $C$  are distinct. So, free to permute the colours, we may assume that  $c_1$  and  $c_2$  agree on  $C$ . One can also choose for  $u$  a colour of  $\{1, \dots, 5\} \setminus \{c_2(x_1), c_2(x_2), c_2(x_3)\}$  so that  $c_2(u) \neq c_1(v)$ . Then the union of  $c_1$  and  $c_2$  is a 5-colouring of  $G$ .  $\square$

LEMMA 2.11. *Let  $G$  be a 6-critical graph. In every drawing of  $G$  in the plane, there is no non-crossed 4-cycle  $C$  such that*

- *$C$  has a chord in its exterior,*
- *$C$  and its interior is a plane graph, and*
- *the interior of  $C$  contains at least one vertex.*

*Proof.* Suppose, by way of contradiction, that there is a 4-cycle  $C = tuvw$  satisfying the properties above with  $vt$  a chord in its exterior. Consider the graph  $G_1$ , which is obtained from  $G$  by removing the vertices inside  $C$ . Since  $G$  is 6-critical,  $G_1$  admits a 5-colouring  $c_1$  in  $\{1, 2, 3, 4, 5\}$ . Without loss of generality, we may assume that  $c_1(v) = 5$ . Hence  $\{c_1(t), c_1(u), c_1(w)\} \subset \{1, 2, 3, 4\}$ .

Now consider the graph  $G_2$  which is obtained from  $G$  by removing the vertices outside  $C$ . If  $c_1(u) = c_1(w)$ , let  $H$  be the graph obtained from  $G_2 - v$  by identifying  $u$  and  $w$ . If  $c_1(u) \neq c_1(w)$ , let  $H$  be the graph obtained from  $G_2 - v$  by adding the edge  $uw$  if it does not already exist. In both cases  $H$  is a planar graph. Hence  $H$  admits a 4-colouring  $c_2$  in  $\{1, 2, 3, 4\}$ . Moreover, by construction of  $H$ ,  $c_2(u) = c_2(w)$  if and only if  $c_1(u) = c_1(w)$ . Hence free to permute the colours, we may assume that  $c_1$  and  $c_2$  agree on  $\{t, u, w\}$ .

Hence the union of  $c_1$  and  $c_2$  is a 5-colouring of  $G$ .  $\square$

**2.3. 6-critical graphs embeddable on the torus or the Klein bottle.** In the proof of Theorem 1.6, we use the list of all 6-critical graphs embeddable on the torus, which was obtained by Thomassen [17], and the list of all 6-critical graphs embeddable on the Klein bottle, which was obtained independently by Chenette et al. [4] and Kawarabayashi et al. [11].

THEOREM 2.12 (Thomassen [17]). *There are four non-isomorphic 6-critical graphs embeddable on the torus. Three of them are depicted in Figure 2.2 and the last one is a 6-regular graph on 11 vertices.*

THEOREM 2.13 (Chenette et al. [4]; Kawarabayashi et al. [11]). *There are nine non-isomorphic 6-critical graphs embeddable on the Klein bottle. They are depicted in Figure 2.2.*

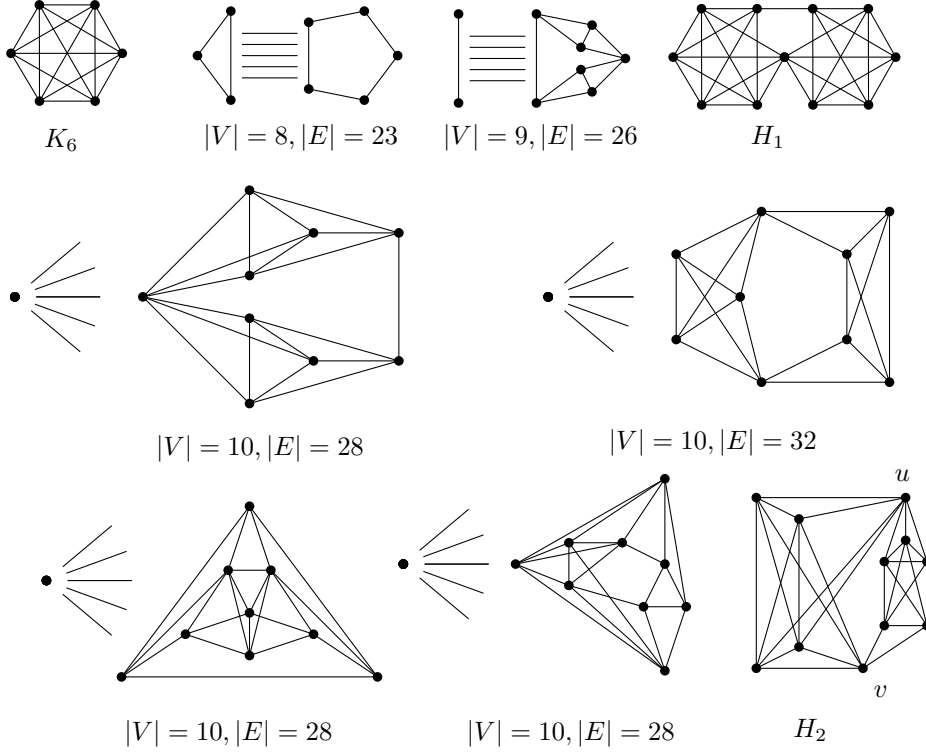


FIG. 2.2. The list of 6-critical graphs embeddable on the Klein bottle. The first three of them are embeddable on torus as well.

LEMMA 2.14. *If three edges are deleted a 6-critical graph embeddable on the torus other than  $K_6$ , then the resulting graph is nonplanar.*

*Proof.* We know the complete list of graphs which must be checked due to Theorem 2.12. For all of them except  $K_6$ , we have  $|E| > 3|V| - 3$ . Thus the graphs are not planar after removing three edges according to Proposition 2.1.  $\square$

LEMMA 2.15. *If three edges are deleted a 6-critical graph embeddable on the Klein bottle other than  $K_6$ , then the resulting graph is nonplanar.*

*Proof.* We know the complete list of graphs which must be checked due to Theorem 2.13. For all of those graphs except  $K_6$ ,  $H_1$  and  $H_2$ , we have  $|E| > 3|V| - 3$ . Thus those graphs are not planar after removing three edges according to Proposition 2.1.

Now we need to deal with the last two graphs  $H_1$  and  $H_2$ , see Figure 2.2. Let us first examine  $H_1$ . It contains two edge disjoint copies of  $K_6$  without one edge. Each of these copies needs at least two edges to be removed by Proposition 2.1, so  $H_1$  needs at least four edges to be removed.

Let us now examine  $H_2$ . Let  $F$  be a set of edges such that  $H_2 \setminus F$  is planar. Let us denote by  $u$  and  $v$  the two vertices of the only 2-cut of  $H_2$ , see Figure 2.2. Observe that  $H_2 - \{u, v\}$  is a disjoint union of  $K_5$  and  $K_4$ . Since  $K_5$  is not planar, one edge  $e$  of this  $K_5$  is in  $F$ . But there is still a  $(u, v)$ -path  $P$  in  $K_5 \setminus e$ . Then the union of the graph induced by  $u$ ,  $v$ , the vertices of the  $K_4$  and the path  $P$  is a subdivision of  $K_6$ . Thus, by Proposition 2.1 for  $K_6$ , at least three of its edges must be in  $F$ . Thus  $|F| \geq 4$ .  $\square$

**3. 6-critical graph of crossing number 5.** We prove Theorem 1.3 by exhibiting a drawing of a 6-critical graph  $G$  using 5 crossings which is not  $K_6$ .

THEOREM 3.1. *The graph  $G$  depicted in Figure 3.1 is 6-critical.*

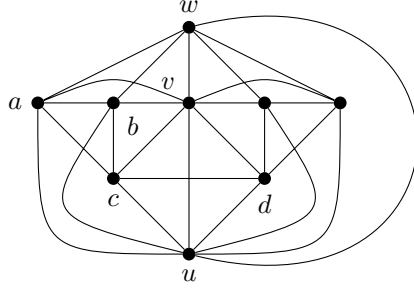


FIG. 3.1. A 6-critical graph of crossing number 5.

*Proof.* We show by contradiction that  $G$  is not 5-colourable. We refer the reader to Figure 3.1 for names of vertices. Assume that  $\varrho$  is a 5-colouring of  $G$ . As vertices  $u, v$  and  $w$  form a triangle, they must get distinct colours. Without loss of generality, assume that  $\varrho(u) = 1, \varrho(v) = 2$  and  $\varrho(w) = 3$ . The vertices  $a$  and  $b$  are adjacent to each other and to all the vertices of the triangle  $uvw$ , hence  $\{\varrho(a), \varrho(b)\} = \{4, 5\}$ . Thus  $\varrho(c) = 3$  as  $c$  is adjacent to  $a, b, u$  and  $v$ . By symmetry we obtain that  $\varrho(d)$  is also 3, which is a contradiction since  $cd$  is an edge.

It can be easily checked that every proper subgraph of  $G$  is 5-colourable. So  $G$  is 6-critical.  $\square$

**4. Colouring graphs whose crossings are covered by few edges.** THEOREM 4.1. *If a graph can be made planar by deleting a set of at most  $2k$  edges, then  $G$  is  $(4+k)$ -colourable.*

*Proof.* We proceed by induction on  $k$ . The result holds when  $k = 0$  by the Four Colour Theorem.

Suppose that the result is true for  $k$ . Let  $G = (V, E)$  be a graph with a set  $F$  of at most  $2k + 2$  edges such that  $G \setminus F$  is planar. Without loss of generality, we may assume that  $F$  is minimal, i.e. for any proper subset  $F' \subset F$ , the graph  $G \setminus F'$  is not planar.

Consider a planar drawing of  $G \setminus F$ . It yields a drawing of  $G$  such that each crossing contains an edge of  $F$ .

Suppose that  $|F| \leq 2k + 1$ . Let  $e = uv$  be an edge of  $F$ . By the induction hypothesis,  $G - v$  is  $(4+k)$ -colourable because  $F \setminus e$  is a set of  $2k$  edges whose removal leaves  $G - v$  planar. Hence  $\chi(G) \leq \chi(G - v) + 1 \leq 4 + k + 1$ .

So we may assume that  $|F| = 2k + 2$ .

If two edges  $e$  and  $f$  of  $F$  have a common vertex  $v$ , then  $G - v$  is  $(4+k)$ -colourable because  $F \setminus \{e, f\}$  is a set of  $2k$  edges whose removal leaves  $G - v$  planar. So  $\chi(G) \leq \chi(G - v) + 1 \leq 4 + k + 1$ . So we may assume that the edges of  $F$  are pairwise non-adjacent.

Let  $e = \{u_1, u_2\}$  and  $f = \{v_1, v_2\}$  be two edges in  $F$ . Then the endvertices of these two edges induce a  $K_4$ . Suppose for contradiction that  $u_1$  and  $v_1$  are not



adjacent. Then  $G - \{u_1, v_1\}$  is  $(4+k)$ -colourable because  $F \setminus \{e, f\}$  is a set of  $2k$  edges whose removal leaves  $G - \{u_1, v_1\}$  planar and  $u_1$  and  $v_1$  can get the same colour. So  $\chi(G) \leq \chi(G - \{u_1, v_1\}) + 1 \leq 4 + k + 1$ . Hence  $X = \{u_1, u_2, v_1, v_2\}$  induces a  $K_4$ .

We further distinguish two possible cases:

- $k = 0$ : Let the edges of  $F$  be  $e = \{u_1, u_2\}$  and  $f = \{v_1, v_2\}$  and let  $X = \{u_1, u_2, v_1, v_2\}$ . Let  $C$  be the 4-cycle induced by  $X$  in the plane graph  $G \setminus \{e, f\}$ . Note that  $C$  is a separating cycle, otherwise  $G \setminus e$  would be planar. We cut  $G$  along  $C$  and obtain two smaller graphs  $G_1$  and  $G_2$ , where both of them contain  $X$ . We 5-colour them by induction. A colouring of  $G$  can be then obtained from the 5-colourings of  $G_1$  and  $G_2$  by permuting colours on  $X$  so that the these two colourings agree on  $V(C)$ .
- $k \geq 1$ : Note that union of all endvertices of edges from  $F$  induce a complete graph  $K_{2|F|}$ . A  $K_{2|F|}$  must be planar after removing at most  $|F|$  edges. Hence the following Euler's formula holds:

$$\begin{aligned} |E| &\leq 3|V| - 6 + 2k + 2 \\ \binom{4k+4}{2} &\leq 3(4k+4) + 2k - 4 \\ 8k^2 - 2 &\leq 0 \end{aligned}$$

Hence this case is not possible.

□

Since  $\text{cr}(K_5) = 1$  and  $\text{cr}(K_6) = 3$ , Theorem 4.1 is tight when  $k \leq 2$ . But  $K_6$  is the only obstacle for pushing the result further as shown by the following theorem which is a reformulation of Theorem 1.6 in terms of critical graphs.

**THEOREM 4.2.** *If at most three edges are deleted from a 6-critical graph other than  $K_6$ , then the resulting graph is nonplanar.*

*Proof.* Let  $G$  be a 6-critical graph distinct from  $K_6$  and  $F$  a set of at most three edges. Assume for a contradiction that  $G \setminus F$  is planar.

Let us consider a nice drawing of  $G$ . By Lemma 2.7,  $G$  has no stable crossing cover.

If  $|F| \leq 2$ , then Theorem 4.1 contradicts the fact that  $G$  is not 5-colourable. Hence we assume that  $F = \{e_1, e_2, e_3\}$ . Set  $e_i = u_i v_i$  for  $i \in \{1, 2, 3\}$ .

**CLAIM 1.** *The three edges of  $F$  are pairwise vertex-disjoint.*

*Proof.* If there is a vertex  $v$  shared by all three edges, then  $\{v\}$  is a stable crossing cover, a contradiction. Hence a vertex  $u$  is shared by at most two edges of  $F$ . Let  $s$  be the number of degree two vertices in the graph induced by  $F$ .

We now derive a contradiction for each value of  $s > 0$ . So  $s = 0$ , which proves the claim.

- $s = 1$ : Without loss of generality,  $u = u_1 = u_2$ . None of  $\{u, u_3\}$  and  $\{u, v_3\}$  is a stable crossing cover so  $uu_3$  and  $uv_3$  are edges. We redraw the edge  $e_3$  along the path  $u_3 uv_3$  such that it crosses only edges incident to  $u$ . See Figure 4.1(A). Then  $u$  is a stable crossing cover, a contradiction.
- $s = 2$ : Without loss of generality,  $u = u_1 = u_2$  and  $v = v_2 = v_3$ . Then  $F$  induces a path. None of  $\{v_1, v\}$  and  $\{u, u_3\}$  is a stable crossing cover, so  $v_1 v$  and  $uu_3$  are edges. We add a handle between vertices  $u$  and  $v$ . Then we draw edges of  $F$  using the handle, see Figure 4.1(B). Hence  $G$  can be embedded on the torus, which is a contradiction to Lemma 2.14.

$s = 3$ : Without loss of generality,  $u = u_1 = u_2$  is one of the shared vertices. Let  $v$  and  $w$  be the other two. Note that  $F$  induces a triangle. By Proposition 2.1, we have  $|E(G)| \leq 3|V(G)| - 3$ . Hence there must be at least 6 vertices of degree five as the minimum degree of  $G$  is five.

Let  $x$  be a degree five vertex different from  $u, v$  and  $w$ . By minimality of  $G$ , there exists a 5-colouring  $\varrho$  of  $G - x$ . Free to permute the colours, we may assume that  $\varrho(u) = 1$ ,  $\varrho(v) = 2$  and  $\varrho(w) = 3$ . Moreover, the neighbours of  $x$  are coloured all differently. We denote by  $y$  and  $z$  the neighbours of  $x$ , which are coloured 4 and 5 respectively. We assume that  $G$  is embedded in the plane such that all crossings are covered by  $F$ . There are two consecutive neighbours of  $x$  in the clockwise order such that they have colours in  $\{1, 2, 3\}$ . We denote these vertices by  $a$  and  $b$ . Without loss of generality let the clockwise order around  $x$  be  $z, y, a, b$  and  $\varrho(a) = 1$  and  $\varrho(b) = 2$ . See Figure 4.1(C).

Let  $A$  be the connected component of  $a$  in the graph induced by the vertices coloured 1 and 5. If  $A$  does not contain  $z$ , we can switch colours on it. Then  $x$  can be coloured by 1 and we have a contradiction. Note that the colour switch is correct even if  $u$  is in  $A$  because the new colour of  $u$  will be 5 which different from 2 and 3. Thus there must be a path between  $a$  and  $z$  of vertices coloured 1 and 5. A similar argument shows that there is a path between  $b$  and  $y$  of vertices coloured 2 and 4. These paths must be disjoint and they are not using edges of  $F$ . But they cannot be drawn in the plane without crossings, a contradiction.

□

CLAIM 2. *For any distinct integers  $i, j \in \{1, 2, 3\}$ , an endvertex of  $e_i$  is adjacent to at most one endvertex of  $e_j$ .*

*Proof.* Suppose not. Then without loss of generality, we may assume that  $u_2$  is adjacent to  $u_1$  and  $v_1$ . First we redraw the edge  $e_1$  along the path  $u_1 u_2 v_1$ . Then every edge crossed by  $e_1$ , which is not  $e_3$ , is incident to  $e_2$ . Since  $\{u_2, u_3\}$  and  $\{u_2, v_3\}$  are not stable crossing covers,  $u_2 u_3$  and  $u_2 v_3$  are edges. We redraw  $e_3$  along the path  $u_3 u_2 v_3$ . Then, again, every edge crossed by  $e_3$ , which is not  $e_1$ , is incident to  $e_2$ . Moreover, the edges  $e_1$  and  $e_3$  cross otherwise  $\{u_2\}$  would be a stable crossing cover. See Figure 4.1(D).

We distinguish several cases according to the number  $p$  of neighbours of  $v_2$  among  $u_1, v_1, u_3$  and  $v_3$ .

$p = 0$ : The vertex  $v_2$  and a pair of two non-adjacent vertices among  $u_1, v_1, u_3$  and  $v_3$  would form a stable crossing cover. Hence  $\{u_1, v_1, u_3, v_3\}$  induces a  $K_4$ . See Figure 4.1(E). By Lemma 2.9, there is no vertex inside each of the triangles  $u_2 u_1 u_3$ ,  $u_2 u_3 v_1$ ,  $u_2 v_1 v_3$  and  $u_2 u_1 v_3$ . Hence all the vertices are inside the 4-cycle  $u_1 u_3 v_1 v_3$ . It includes the vertex  $v_2$ . We redraw  $e_1$  such that it is crossing only  $e_3$  and  $u_2 v_3$ . Then  $\{v_3, v_2\}$  is a stable crossing cover, a contradiction. See Figure 4.1(F).

$p = 1$ : Without loss of generality we may assume that the neighbour of  $v_2$  is  $u_1$ . None of  $\{v_2, v_1, u_3\}$  and  $\{v_2, v_1, v_3\}$  is a stable crossing cover so  $u_3 v_1$  and  $v_1 v_3$  are edges. By Lemma 2.9, there is no vertex inside each of the triangles  $u_2 u_3 v_1$  and  $u_2 v_1 v_3$ . See Figure 4.1(G). Thus the edge  $e_3$  could be drawn inside these triangles and the set  $F$  can be changed to  $F' = \{e_1, e_2, u_2 v_1\}$ . Two edges of  $F'$  share an endvertex which is a contradiction to Claim 1.

$p \in \{2, 3\}$ : We further distinguish two sub-cases. Either two neighbours of  $v_2$  in  $\{u_1, v_1, u_3, v_3\}$  are joined by an edge of  $F$  or not.

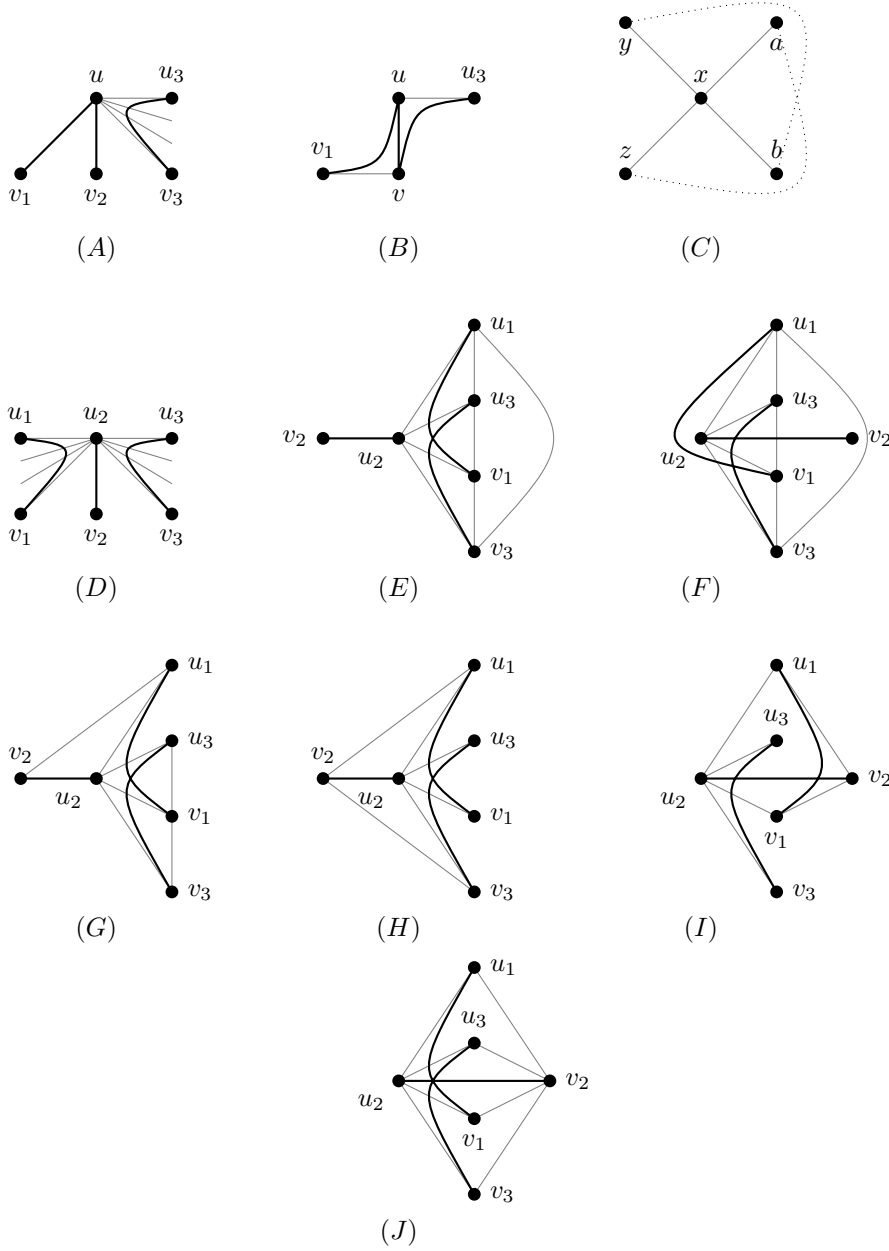


FIG. 4.1. The three black edges are covering all the crossings.

In the second case, without loss of generality, we may assume that the vertices adjacent to  $v_2$  are  $u_1$  and  $v_3$ . Now by Lemma 2.11 there is no vertex inside the 4-cycle  $v_2u_1u_2v_3$ . Hence  $e_2$  can be drawn inside this cycle. See Figure 4.1(H). Since the removal of  $\{e_1, e_3\}$  does not make  $G$  planar,  $v_1v_3$  is inside  $v_2u_1u_2v_3$ . Hence the set  $F' = \{e_1, e_3, u_1v_3\}$  contradicts Claim 1.

In the first case, we may assume Without loss of generality, that  $v_2$  is adjacent

to  $u_1$  and  $v_1$ . We first redraw  $e_1$  along the path  $u_1v_2v_1$ . Now all the edges crossing  $e_1$  are incident to  $v_2$ . Thus  $\{v_2, u_3\}$  or  $\{v_2, v_3\}$  form a stable crossing cover. See Figure 4.1(I).

$p = 4$ : See Figure 4.1(J). We repeatedly use Lemma 2.11 which implies that the 4-cycles  $u_2u_3v_2u_1$ ,  $u_2u_3v_2v_1$ ,  $u_2v_1v_2v_3$  and  $u_2v_3v_2u_1$  are not separating. This means that the graph contains only six vertices. This is a contradiction because the unique 6-critical graph on six vertices is  $K_6$ .

□

Since  $\{u_1, u_2, u_3\}$  is not a stable crossing cover, it must induce at least one edge, say  $u_1u_2$ . Then Claim 2 implies that  $u_1v_2$  and  $v_1u_2$  are not edges. Now  $\{v_1, u_2, u_3\}$  and  $\{v_1, u_2, v_3\}$  are not stable crossing covers. Thus, by symmetry, we may assume that  $u_2u_3$  and  $v_1v_3$  are edges.  $\{u_1, v_2, u_3\}$  is not a stable crossing cover so  $u_1u_3$  is an edge;  $\{v_1, v_2, u_3\}$  is not a stable crossing cover so  $v_1v_2$  is an edge;  $\{u_1, v_2, v_3\}$  is not a stable crossing cover so  $v_2v_3$  is an edge. Hence there are two triangles  $u_1u_2u_3$  and  $v_1v_2v_3$ , which are not separating by Lemma 2.9.

Without loss of generality, two possibilities occur. Either the edges of  $F$  do not cross each other or one pair of them is crossing. If they do not cross (Figure 4.2(A)),  $G$  can be embedded on the torus by adding a handle into the triangles and drawing the edges of  $F$  on the handle, which contradicts Lemma 2.14.

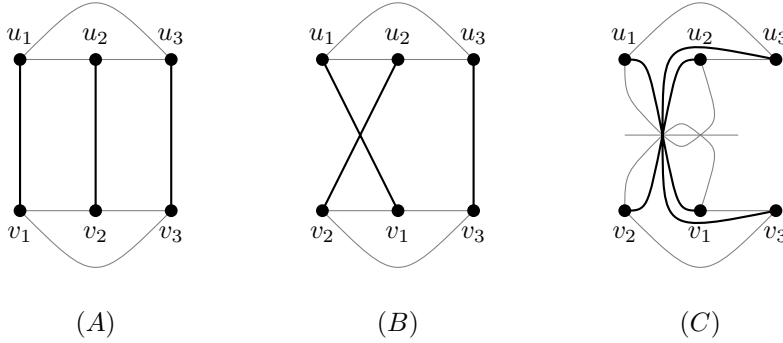


FIG. 4.2. The last case of Theorem 1.6.

If they cross (Figure 4.2(B)), it is possible to draw  $G$  on the Klein bottle, see Figure 4.2(C), which contradicts Lemma 2.15. □

**5. 5-colouring graphs with 4 crossings.** In this section we prove the following Theorem 5.1, which is a reformulation of Theorem 1.4 in terms of critical graphs.

**THEOREM 5.1.** *The unique 6-critical graph with crossing number at most 4 is  $K_6$ .*

*Proof.* Suppose, by way of contradiction, that  $G = (V, E)$  is a 6-critical graph with crossing number at most 4 distinct from  $K_6$ . Moreover, one may assume that  $G$  is such a critical graph with the minimum number of vertices and with the maximum number of edges on  $|V(G)|$  vertices.

Moreover, assume that we have a nice optimal drawing of  $G$ . By Theorem 1.6, there are four crossings and every edge is crossed at most once.

Since  $G$  is 6-critical, every vertex has degree at least 5. By Proposition 2.1, we have  $|E| \leq 3|V| - 6 + \text{cr}(G) \leq 3|V| - 2$ . Hence there are at least four vertices of degree 5.

Let  $v$  be an arbitrary degree five vertex and  $v_i$ ,  $1 \leq i \leq 5$  be the neighbours of  $v$  in the counterclockwise order around  $v$ . By criticality of  $G$ , the graph  $G - v$  admits a 5-colouring  $\phi$ . Necessarily, all the  $v_i$  are coloured differently, otherwise  $\phi$  could be extended to  $v$ .

For any  $i \leq j$ , there is a path, denoted by  $v_i - v_j$ , from  $v_i$  to  $v_j$  such that all its vertices are coloured in  $\phi(v_i)$  or  $\phi(v_j)$ . Otherwise,  $v_j$  is not in the connected component  $A$  of  $v_i$  in the graph induced by the vertices coloured  $\phi(v_i)$  and  $\phi(v_j)$ . Hence by exchanging the colours  $\phi(v_i)$  and  $\phi(v_j)$  on  $A$ , we obtain a 5-colouring  $\phi'$  of  $G - v$  such that no neighbour of  $v$  is coloured  $\phi(v_i)$ . Hence by assigning  $\phi(v_i)$  to  $v$  we obtain a 5-colouring of  $G$ , a contradiction.

Let  $q$  be the number of crossed edges incident to  $v$ .

CLAIM 3.  $q \neq 0$ .

*Proof.* The union of the  $v_i - v_j$ , for  $i \neq j$ , is a subdivision of  $K_5$  in  $G - v$ . If  $q = 0$ , then the  $v_i$ ,  $1 \leq i \leq 5$ , are in one face after the removal of  $v$ . By Lemma 2.5, such a subdivision requires 5 crossings which contradicts the assumption of at most four crossings.  $\square$

CLAIM 4.  $q \neq 1$ .

*Proof.* Suppose to the opposite that  $q = 1$ . Without loss of generality, we may assume that the crossed edge is  $vv_1$ .

The path  $v_2 - v_4$  must cross the two paths  $v_1 - v_3$  and  $v_3 - v_5$ . Since every edge is crossed at most once, then  $v_2v_4$  is not an edge.

Let  $G'$  be the graph obtained from  $G - v$  by identifying  $v_2$  and  $v_4$  into a new vertex  $v'$ . By Lemma 2.8,  $G'$  is not 5-colourable. Now  $G'$  has at most three crossings because we removed the crossed edge  $vv_1$  together with  $v$ . So, by minimality of  $G$ , the graph  $G'$  contains a subgraph  $H$  isomorphic to  $K_6$ . Moreover,  $H$  must contain  $v'$  since  $G$  contains no  $K_6$ . Since  $G'$  has only three crossings we can use Lemma 2.4. Let  $u_1$  and  $u_2$  be vertices of  $H$  which form a triangular face together with  $v'$  and let  $u_3, u_4$  and  $u_5$  be the vertices forming the other triangular face. Without loss of generality, we may assume that  $u_3u_4u_5$  is inside  $v'u_1u_2$  as in Figure 5.1(A).

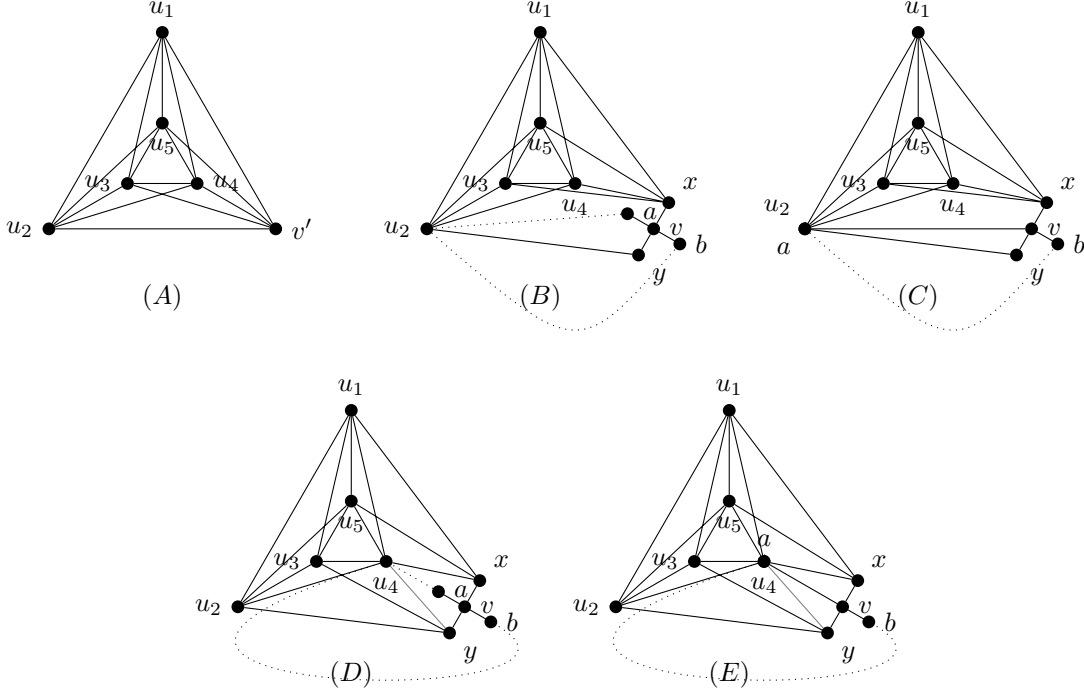
Let us now consider the situation in  $G$ . Instead of discussing many rotations of  $K_6$  we rather fix  $K_6$  and try to investigate possible placings of  $v$  and its neighbours. We denote the neighbours of  $v$  which were identified by  $x$  and  $y$  (i.e.  $\{v_2, v_4\} = \{x, y\}$ ). Let  $a$  and  $b$  be the two other neighbours of  $v$  such that  $va$  and  $vb$  are not crossed ( $\{a, b\} = \{v_3, v_5\}$ ). Moreover, we assume that in the counterclockwise order around  $v$ , the sequence is  $x, a, y, b$ . Note that the vertex  $v_1$  may be inserted anywhere in the sequence.

One of the identified vertices, say  $x$ , is adjacent to at least two vertices of  $\{u_3, u_4, u_5\}$ .

- 1) Assume first that  $x$  is adjacent to  $u_3, u_4$  and  $u_5$ . Then since  $G$  has no  $K_6$ , it is not adjacent to some vertex in  $\{u_1, u_2\}$ , say  $u_2$ . Thus  $yu_2 \in E$ .

The vertex  $a$  is either inside  $u_2yv_x$  or is  $u_2$ . See Figure 5.1(B) and (C), respectively. The path  $a - b$  (represented by dotted line in the figure) necessarily uses  $u_2$ . Since colours  $\phi(a)$  and  $\phi(b)$  alternate on  $a - b$ , this path cannot contain  $x$  nor  $u_3, u_4$  and  $u_5$ . The paths  $a - b$  and  $avb$  separate  $x$  and  $y$  and there must be paths  $v_1 - x$  and  $v_1 - y$ . Thus at least one of them must cross the path  $a - b$ . But none of the four crossings is available for that, a contradiction.

- 2) Let us now assume that  $x$  is adjacent to only two vertices of  $\{u_3, u_4, u_5\}$ , say

FIG. 5.1.  $K_6$  when identifying two neighbours of  $v$ .

$u_4$  and  $u_5$ . Then  $u_3$  is adjacent to  $y$ . (Possibly  $u_4$  and  $y$  are adjacent too.) The path  $a - b$  must go through  $u_4$  and then continue to  $u_1$  or  $u_2$ . It cannot go through  $u_3$  or  $u_5$  since the colours on the path alternate. See Figure 5.1(D) and (E).

The path  $x - y$  must cross  $a - b$ . Hence either  $x - y$  goes through  $u_3y$  and  $a - b$  through  $u_4u_2$  or  $x - y$  goes through  $xu_5$  and  $a - b$  through  $u_4u_1$ . In both cases, one of the paths  $v_1 - x$  and  $v_1 - y$  must cross  $a - b$ . But there are no more crossings available.

This completes the proof of Claim 4.  $\square$

CLAIM 5.  $q \neq 2$ .

*Proof.* Suppose to the opposite that  $q = 2$ .

We first prove the following assertion that will be used several times.

**Assertion** *Let  $x$  and  $y$  be two neighbours of  $v$ . Then  $x$  and  $y$  are adjacent if one of the following holds:*

- $vx$  and  $vy$  are not crossed;
- $\{x, y\}$  is included in the cluster of some crossing.

Observe that  $G - v$  has at most two crossings. Suppose that  $x$  and  $y$  are not adjacent. If  $vx$  and  $vy$  are not crossed, we can identify  $x$  and  $y$  along  $xvy$  without adding any new crossing. If  $\{x, y\}$  is included in the cluster of some crossing, we can identify  $x$  and  $y$  along the edges of this crossing without adding any new crossing. Hence in both cases  $(G - v)/\{x, y\}$  has a planar drawing with at most 2 crossings. Then Lemma 2.8 and Theorem 4.1 yield a contradiction. This proves the Assertion.

Assume that the crossed edges are consecutive, say  $vv_1$  and  $vv_2$ . By the Assertion,  $v_3v_5$  is an edge. See Figure 5.2(A). If  $v_3v_5$  is not crossed or crosses either  $vv_1$  and  $vv_2$ , then the cycle  $vv_3v_5$  is regular, which contradicts Lemma 2.9. If  $v_3v_5$  is crossed by another edge, then the cycle  $vv_3v_5$  contradicts Lemma 2.10. Henceforth, we may assume that the two crossed edges are not consecutive, say  $vv_2$  and  $vv_5$ .

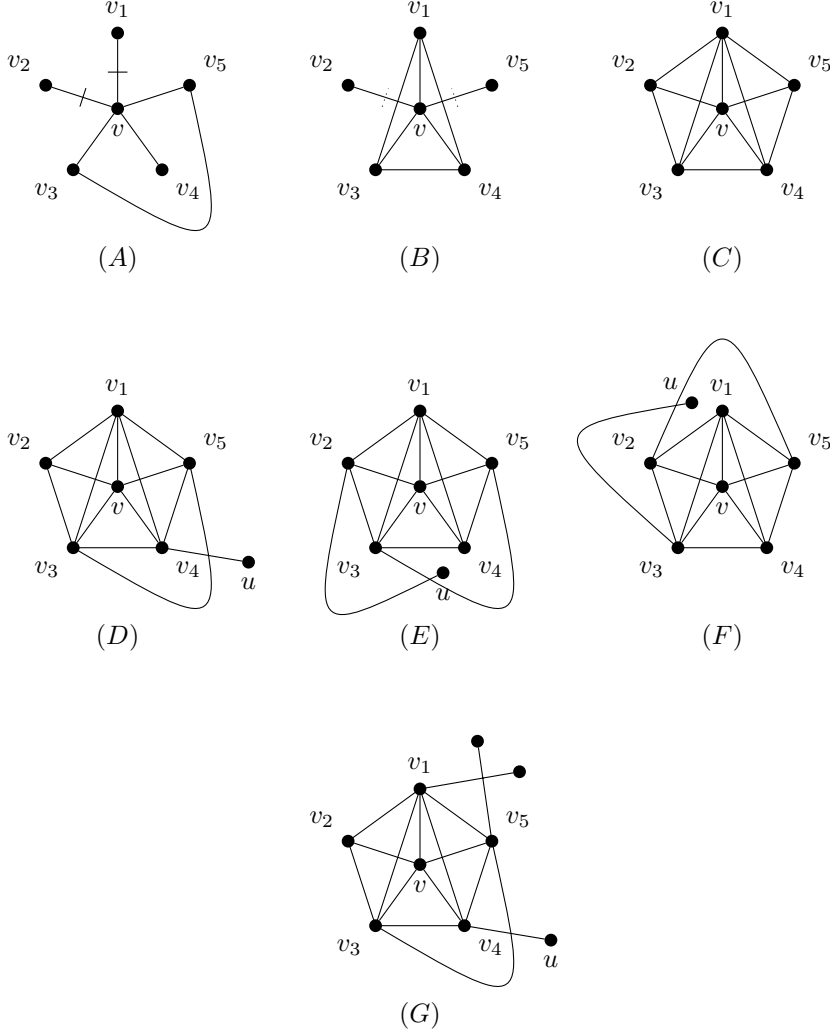


FIG. 5.2. Two crossed edges.

By the Assertion,  $v_1v_3$ ,  $v_1v_4$  and  $v_3v_4$  are edges. If  $v_1v_3$  is not crossed, then the triangle  $vv_1v_3$  is separating because  $v_2$  and  $v_4$  are on the opposite sides. This contradicts Lemma 2.9. If  $v_1v_3$  is crossed it can be redrawn along the path  $v_1vv_3$  with one crossing with  $vv_2$ . Symmetrically, we assume that  $v_1v_4$  is crossing  $vv_5$ . See Figure 5.2(B).

By the Assertion,  $\{v_1v_2, v_2v_3, v_4v_5, v_5v_1\} \subset E(G)$ . See Figure 5.2(C).

Let  $C = \{c_1c_2, c_3c_4\}$  and  $D = \{d_1d_2, d_3d_4\}$  be the two crossings not having  $v$  in their cluster. For convenience and with a slight abuse of notation, we denote by

$C$  (resp.  $D$ ) both the crossing  $C$  (resp.  $D$ ) and its cluster. For  $X \in \{C, D\}$ , let  $a(X) := |X \cap N(v)|$ . Without loss of generality, we may assume that  $a(C) \leq a(D)$ .

A vertex  $u$  is a *candidate* if it is not adjacent to  $v$ . There is no candidate  $u$  common to both  $C$  and  $D$  otherwise  $\{u, v\}$  would be a stable crossing cover. There are no non-adjacent candidate vertices  $c \in C$  and  $d \in D$  otherwise  $\{v, c, d\}$  would be a stable crossing cover.

Assume that  $a(D) = 4$ . The vertex  $v_1$  cannot be in  $D$  because it is already adjacent to all the other neighbours of  $v$  by edges not in  $D$ . Thus  $D = \{v_2, v_3, v_4, v_5\}$ . But then, by the Assertion,  $v_2v_5$  is an edge. So  $N(v) \cup \{v\}$  induces a  $K_6$ , a contradiction. Hence  $a(C) \leq a(D) \leq 3$ .

Suppose now that  $X \in \{C, D\}$  does not induce a  $K_4$ . Then two vertices  $x_1$  and  $x_2$  of  $X$  are not adjacent. One can add the edge  $x_1x_2$  and draw it along the edges of the crossing such that no new crossing is created. Hence by the choice of  $G$ , the obtained graph  $G \cup x_1x_2$  contains a  $K_6$ . Since  $K_6$  has crossing number 3, one of the crossings containing  $v$  in its cluster must be used. So  $v$  belongs to the  $K_6$  and hence the  $K_6$  is induced by  $\{v\} \cup N(v)$ . In such case edges  $v_2v_4$  and  $v_3v_5$  cross and hence form  $C$  or  $D$ , which is not possible since  $a(C) \leq a(D) \leq 3$ .

Hence both  $C$  and  $D$  induce a  $K_4$ . Thus the candidates in  $C \cup D$  induce a complete graph. So there are at most five of them. Since  $C \cap D$  contains no candidate, we have  $a(C) + a(D) \geq 3$  and so  $2 \leq a(D) \leq 3$ .

Assume that  $a(D) = 2$  and thus  $1 \leq a(C) \leq 2$ . Then  $C$  (resp.  $D$ ) contains a set  $C'$  (resp.  $D'$ ) of two candidates. All the vertices of  $C'$  are adjacent to all the vertices of  $D'$ . But since both  $C$  and  $D$  contain a vertex in  $N(v)$ , drawing all the edges between these two sets requires one more crossing, a contradiction. Hence  $a(D) = 3$ .

Thus, an edge of  $D$  has its two endvertices in  $N(v)$  and so it is  $v_2v_5$ ,  $v_2v_4$  or  $v_3v_5$ . Let  $u$  be the unique candidate of  $D$ .

Assume first that  $v_1 \in D$ . Then  $v_1u$  is an edge of  $D$ . Moreover,  $C$  must be on the paths  $v_2 - v_4$  and  $v_3 - v_5$ . Since edges are crossed at most once  $D = \{v_1u, v_2v_5\}$ . Let  $w$  be a candidate vertex in  $C$ . Then  $w$  is outside the cycle  $vv_2v_5$ . But the only neighbour of  $v_1$  outside this cycle is  $u$  which is distinct from  $w$  because the crossings  $C$  and  $D$  have no candidate vertex in common. Thus  $\{w, v_1\}$  is a stable crossing cover, a contradiction to Lemma 2.7.

So  $v_1 \notin D$ .

By symmetry, we may assume that  $D$  is either  $\{v_3v_5, v_4u\}$  (Figure 5.2(D)) or  $\{v_3v_5, v_2u\}$  (Figure 5.2(E)) or  $\{v_2v_5, v_3u\}$  (Figure 5.2(F)). In the second and third cases, Lemma 2.10 is contradicted by the cycle  $v_3v_4v_5$  and  $v_1v_2v_5$  respectively. Hence  $D = \{v_3v_5, v_4u\}$ .

The set  $\{v_2, v_4\}$  is stable and covers the three crossings distinct from  $C$ . Hence  $\{v_2, v_4\}$  does not intersect  $C$ , otherwise it would be a stable crossing cover. So  $C \cap N(v) \subset \{v_1, v_3, v_5\}$ . The edge  $v_1v_5$  is not crossed, otherwise it could be redrawn along the edges of the crossing  $\{vv_5, v_1v_4\}$  to obtain a drawing of  $G$  with fewer crossings. Furthermore,  $v_1v_3$  and  $v_3v_5$  are not in  $C$  because they are in some other crossing. Hence  $a(C) \leq 2$ .

Let  $B$  be the set of candidates of  $C$ . Recall that all vertices of  $B$  are adjacent to  $u$ . Moreover, every vertex  $b \in B$  is adjacent to a vertex of  $\{v_2, v_4\}$  otherwise  $\{v_2, v_4, b\}$  is a stable crossing cover. But  $v_4$  and  $u$  are separated by  $v_3v_4v_5$ , so all vertices of  $B$  are adjacent to  $v_2$ . Now the graph induced by the edges between  $B$  and  $\{u, v_2\}$  is a complete bipartite graph. Moreover, its induced drawing has no crossing and the



vertices of each part are in a common face. Thus, by Lemma 2.6,  $|B| \leq 2$ . So  $a(C) = 2$ .

Recall that  $C \cap N(v) \subset \{v_1, v_3, v_5\}$ . Suppose that  $C \cap N(v) = \{v_1, v_3\}$ . The closed curve formed by the path  $v_3vv_1$  and the two “half-edges” connecting  $v_1$  to  $v_3$  in  $C$  separates  $v_2$  and  $u$ . Then the vertices of  $B$  cannot be adjacent to both  $u$  and  $v_2$ , a contradiction. Similarly, we obtain a contradiction if  $C \cap N(v) = \{v_3, v_5\}$ . Hence we may assume that  $C \cap N(v) = \{v_1, v_5\}$ . But then connecting the vertices of  $B$  to those of  $\{v_2, v_4\}$  would require one more crossing. See Figure 5.2(G).

This completes the proof of Claim 5.  $\square$

CLAIM 6.  $q \neq 3$ .

*Proof.* Suppose that  $q = 3$ .

Let  $C$  be the crossing whose cluster does not contain  $v$ . It contains no candidate  $u$  otherwise  $\{u, v\}$  would be a stable crossing cover. Hence  $C \subset N(v)$ .

Assume first that the three crossed edges incident to  $v$  are consecutive, say the crossed edges are  $vv_1, vv_2$  and  $vv_5$ . By the Assertion,  $v_3v_4$  is an edge. See Figure 5.3(A). Up to symmetry, the cluster of  $C$  is one of the following three sets  $\{v_1, v_2, v_3, v_4\}$  or  $\{v_2, v_3, v_4, v_5\}$  or  $\{v_1, v_2, v_4, v_5\}$ .

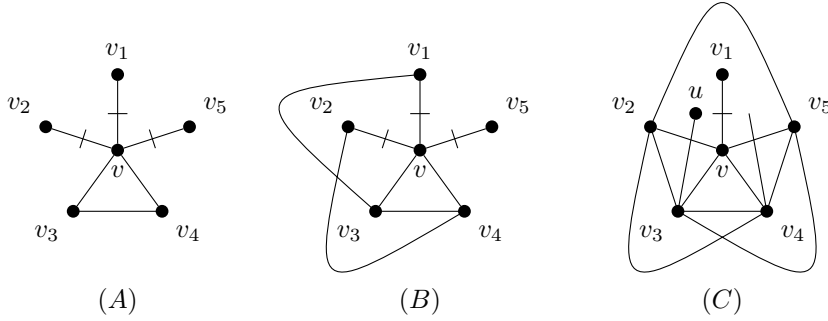


FIG. 5.3. Three consecutive crossed edges.

- $C = \{v_1, v_2, v_3, v_4\}$ . Then the edges of  $C$  are not  $v_1v_4$  and  $v_2v_3$  because it is impossible to draw them such that each is crossed exactly once. Hence  $C = \{v_1v_3, v_2v_4\}$ . The Jordan curve formed by the path  $v_1vv_4$  and the two “half-edges” connecting  $v_1$  to  $v_4$  in  $C$  separates  $\{v_2, v_3\}$  and  $v_5$ . See Figure 5.3(B). Moreover, it is crossed only once (on edge  $v_1v$ ), while two crossings are needed, one for each of the disjoint paths  $v_2 - v_5$  and  $v_3 - v_5$ , a contradiction.
- $C = \{v_2, v_3, v_4, v_5\}$ . Then the edges of  $C$  are not  $v_2v_3$  and  $v_4v_5$  because it is impossible to draw them such that each is crossed exactly once. Hence  $C = \{v_2v_4, v_3v_5\}$ . Hence by the Assertion,  $v_2v_3$ ,  $v_4v_5$  and  $v_2v_5$  are edges. The triangle  $vv_2v_3$  has only one crossed edge. So, by Lemma 2.10, it is not separating. Thus its interior is empty and the edge crossing  $vv_2$  is incident to  $v_3$ . Let  $u$  be the second endvertex of this edge. By symmetry, the interior of  $vv_4v_5$  is empty and the edge crossing  $vv_5$  is  $v_4t$  for some vertex  $t$ . If  $u = t = v_1$ , then by the Assertion  $v_1v_2$  and  $v_1v_5$  are edges. So  $N(v) \cup \{v\}$  induces a  $K_6$ , a contradiction. Hence without loss of generality we may assume that  $u \neq v_1$ . See Figure 5.3(C).

The interiors of the cycles  $vv_2v_3$ ,  $vv_3v_4$  and  $v_2v_3v_4$  contain no vertices by Lemma 2.9. Hence  $v_3$  is a degree five vertex. Moreover, its two neighbours  $u$  and  $v$  are not adjacent and  $(G - v_3)/\{u, v\}$  has at most two crossings. Then Theorem 4.1 and Lemma 2.8 yield a contradiction.

- $C = \{v_1, v_2, v_4, v_5\}$ . The crossing  $C$  is neither  $\{v_1v_2, v_4v_5\}$  nor  $\{v_1v_5, v_2v_4\}$  since it is impossible to draw so that every edge is crossed exactly once. Hence  $C = \{v_1v_4, v_2v_5\}$ . By the Assertion,  $v_2v_4 \in E(G)$ . Then the triangle  $vv_2v_4$  contradicts Lemma 2.10.

Suppose now that the three crossed edges incident to  $v$  are not consecutive. Without loss of generality, we assume that these edges are  $vv_1$ ,  $vv_3$  and  $vv_4$ .

By the Assertion,  $v_2v_5$  is an edge. If  $v_2v_5$  is not crossed, then  $vv_2v_5$  is a separating triangle, contradicting Lemma 2.9. So  $v_2v_5$  is crossed. It could not cross  $vv_3$  nor  $vv_4$  otherwise  $vv_2v_5$  would be a regular cycle contradicting Lemma 2.9. Moreover,  $v_2v_5$  cannot be in  $C$  otherwise  $vv_2v_5$  would contradict Lemma 2.10. Hence  $v_2v_5$  crosses  $vv_1$ .

By the Assertion,  $v_1v_2$  and  $v_1v_5$  are edges. Moreover they are not crossed, otherwise they could be redrawn along the edges of the crossing  $\{vv_1, v_2v_5\}$  to obtain a drawing of  $G$  with fewer crossings. See Figure 5.4(A).

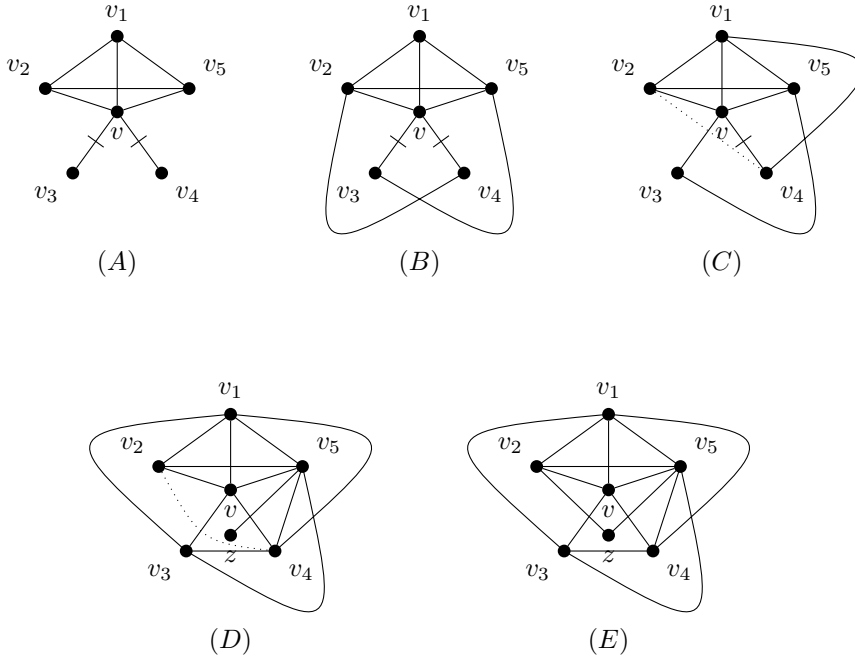


FIG. 5.4. *Three non-consecutive crossed edges.*

Consider the paths  $v_2 - v_4$  and  $v_3 - v_5$ . If they cross, it is through  $C$ . Since  $C \subset N(v)$ , the paths  $v_2 - v_4$  and  $v_3 - v_5$  are actually edges. See Figure 5.4(B). But one can redraw  $v_2v_5$  along the edges of  $C$  to obtain a drawing of  $G$  with fewer crossings, a contradiction.

Suppose now that  $v_2 - v_4$  and  $v_3 - v_5$  do not cross. By symmetry, we may assume that  $v_2 - v_4$  cross  $vv_3$ . The paths  $v_1 - v_4$  and  $v_3 - v_5$  cross. It must be through  $C$  so  $v_1v_4$  and  $v_3v_5$  are both edges. See Figure 5.4(C). By the Assertion,  $v_1v_3$ ,  $v_3v_4$  and

$v_4v_5$  are edges.

If  $v_2v_4$  is also an edge, the Assertion implies that  $v_2v_3$  is also an edge. Then  $N(v) \cup \{v\}$  induces a  $K_6$ , a contradiction. Hence  $v_2v_4 \notin E(G)$ .

By Lemma 2.10, the cycle  $vv_4v_5$  is not separating, so its interior contains no vertex and  $vv_4$  is crossed by an edge with  $v_5$  as an endvertex. Let  $z$  be the other endvertex of this edge. As an edge is crossed at most once,  $z$  is inside  $vv_3v_4$ . See Figure 5.4(D).

Let  $ab$  be the edge which is crossing  $vv_3$ . The sets  $\{v_5, a\}$  and  $\{v_5, b\}$  are not stable otherwise they would be a stable crossing cover. Hence  $v_5a$  and  $v_5b$  are both edges. Thus  $ab = v_2z$ . See Figure 5.4(E). Now  $v_1z$  is not an edge and hence  $\{v_1, z\}$  is a stable crossing cover, contradicting Lemma 2.7.

This completes the proof of Claim 6.  $\square$

CLAIM 7.  $q \neq 4$

*Proof.* By way of contradiction, suppose that  $q = 4$ . Then  $\{v\}$  is a stable crossing cover, a contradiction.  $\square$

Combining Claims 3, 4, 5, 6 and 7 yields a contradiction. This finishes the proof of Theorem 5.1.  $\square$

## 6. Further research.

**6.1. Extending our results.** Theorem 4.1 states that, if a graph can be made planar by removing at most  $2k$  edges, then it is  $(4 + k)$ -colourable. We believe that this is not tight. Thus a natural question is the following:

PROBLEM 6.1. *What is the maximum chromatic number of all graphs that contain a set of at most  $k$  edges whose deletion results in a planar graph?*

Let us denote this maximum by  $g(k)$ . Clearly,  $g(1) = g(2) = 5$  by Theorem 4.1 and because  $K_5$  is not planar and  $g(3) = 6$  by Theorem 4.1 and because  $cr(K_6) = 3$ . For larger value of  $k$ , we also believe that the optimal value is given by a complete graph. It is also very likely that the complete graph  $K_{g(k)}$  is the unique  $g(k)$ -critical graphs that can be made planar by removing  $k$  edges. It is in particular the case for  $k = 6$  and  $k = 7$ . Indeed by Proposition 2.1, at least 6 edges are needed to make  $K_7$  planar and there is a set of 6 edges whose removal leaves  $K_7$  planar. See Figure 6.1.

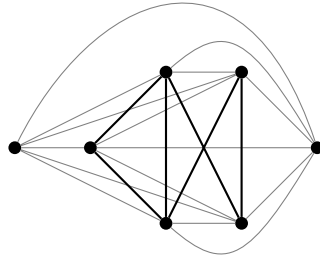


FIG. 6.1. The graph  $K_7$  and a set of edges (in bold) whose removal yields a planar graph.

THEOREM 6.2. *Let  $G$  be a graph with clique number at most 6. If there is a set of at most 7 edges whose deletion results in a planar graph, then  $G$  is 6-colourable.*

*Proof.* To prove this theorem, we show that  $K_7$  is the unique 7-critical graph  $G$  for which there exists a set of at most 7 edges whose removal leaves  $G$  planar. A

famous result of Dirac [5] states that, if  $G$  is  $r$ -critical graph and is not  $K_r$ , then  $2|E(G)| \geq (r-1)|V(G)| + r - 3$ . In particular, if  $r = 7$ , then  $|E(G)| \geq 3|V(G)| + 2$ . Hence by Proposition 2.1, we need to remove at least 8 edges to make it planar.  $\square$

One of the first problem to tackle is the following conjecture which extends both Theorem 1.6 and Theorem 1.4.

**CONJECTURE 6.3.** *Every graph with clique number at most 5 and that contains a set of at most four edges whose deletion results in a planar graph is 5-colourable.*

**6.2. Critical graphs and colourability.** It is easy to derive from Proposition 2.1 that for  $r \geq 8$ , there are only finitely many  $r$ -critical graphs that can be embedded on a fixed surface. As pointed out by Thomassen in [17], the number of 7-critical graphs that can be embedded on a fixed surface is also finite. Finally, Thomassen [18] completed the results by showing that the number of 6-critical subgraphs is finite for any fixed surface  $\Sigma$ . This implies in particular that the  $(r-1)$ -colourability problem for graphs embeddable on  $\Sigma$  is decidable in polynomial time for any  $r \geq 6$ . On the other hand, deciding 3-colourability is NP-complete for planar graphs (see [8]) and thus also for graphs embeddable on any other surface. The complexity of 4-colourability remains open.

**PROBLEM 6.4.** *For any fixed surface, does there exists a polynomial time algorithm for deciding if a graph embeddable on this surface is 4-colourable?*

The answer to Problem 6.4 is only known for the sphere by the Four Colour Theorem. An affirmative answer cannot be obtained in the same way as for  $r-1 \geq 5$  because there are infinitely many 5-critical graphs as implied by a result of Fisk [7].

If  $\text{cr}(G) = k$ , then  $G$  is embeddable in  $\mathbb{S}_k$  and in  $\mathbb{N}_k$  as well, where  $\mathbb{S}_k$  is an orientable surface of genus  $k$  and  $\mathbb{N}_k$  is a non-orientable surface of genus  $k$ . Hence for any  $k$  and  $r \geq 6$ , the number of  $r$ -critical graphs of crossing number  $k$  is finite and so the  $(r-1)$ -colourability problem for graphs of crossing number  $k$  is decidable in polynomial time. However, the design of such a polynomial time algorithm requires the knowledge of the list of 6-critical graphs.

**PROBLEM 6.5.** *Let  $k \geq 0$ . What is the list of 6-critical graphs with crossing number at most  $k$ ?*

When  $k \leq 3$ , the list is empty and if  $k = 4$ , then the list is  $\{K_6\}$ . If  $k = 5$ , then the list contains  $K_6$  and the graph depicted in Figure 3.1. But are there any other?

Similarly to graphs embeddable on a fixed surface, the complexity of 4-colourability problem for graphs with crossing number  $k$  is not known.

**PROBLEM 6.6.** *Let  $k \geq 1$ . Does there exists a polynomial time algorithm for deciding if a given graph with crossing number  $k$  is 4-colourable?*

This would be true if there were a finite number 5-critical graphs with crossing number at most  $k$ . However, as observed by Zdeněk Dvořák, this is not the case. Indeed, every graph obtained from an odd cycle by adding two new vertices adjacent to all the vertices of the cycle and adjacent to each other is 5-critical.

**6.3. Choosability.** A *list assignment* of a graph  $G$  is a function  $L$  that assigns to each vertex  $v \in V(G)$  a list  $L(v)$  of available colours. An  *$L$ -colouring* is a function  $\varphi : V(G) \rightarrow \bigcup_v L(v)$  such that  $\varphi(v) \in L(v)$  for every  $v \in V(G)$  and  $\varphi(u) \neq \varphi(v)$  whenever  $u$  and  $v$  are adjacent vertices of  $G$ . If  $G$  admits an  $L$ -colouring, then it is  *$L$ -colourable*. A graph  $G$  is  *$k$ -choosable* if it is  $L$ -colourable for every list assignment  $L$  such that  $|L(v)| \geq k$  for all  $v \in V(G)$ . The *choose number* of  $G$ , denoted by  $\text{ch}(G)$ , is the minimum  $k$  such that  $G$  is  $k$ -choosable.

Similarly to the chromatic number, one may seek for bounds on the choose number of a graph with few crossings or with independent crossings.

Thomassen [16] showed that every planar graph is 5-choosable. In fact, he proved a stronger result.

**DEFINITION 6.7.** An *inner triangulation* is a plane graph such that every inner face of  $G$  is bounded by a triangle and its outer face by a cycle  $F = (v_1 v_2 \dots v_k v_1)$ .

A list assignment  $L$  of an inner triangulation  $G$  is *suitable* if

- $|L(v_1)| = 1$  and  $|L(v_2)| = 2$ ,
- for every  $v \in V(F) \setminus \{v_1, v_2\}$ ,  $|L(v)| \geq 3$ , and
- for every  $v \in V(G) \setminus V(F)$ ,  $|L(v)| \geq 5$ .

**THEOREM 6.8** (Thomassen [16]). *If  $L$  is a suitable list assignment of an inner triangulation  $G$ , then  $G$  is  $L$ -colourable.*

**THEOREM 6.9.** *Every graph with crossing number 1 is 5-choosable.*

*Proof.* Let  $G$  be a graph with crossing number 1. Consider a plane embedding of  $G$  with one crossing  $C = \{x_1 y_1, x_2 y_2\}$ . Without loss of generality, we may assume that  $C$  is in the outer face. Free to add edges, we may assume that the outer face is bounded by the 4-cycle  $x_1 x_2 y_1 y_2 x_1$  and that  $G \setminus C$  is an inner triangulation.

Let  $L$  be a 5-list assignment of  $G$ . Set  $c_1 \in L(x_1)$  and  $c_2 \in L(x_2) \setminus \{c_1\}$ . Let  $L'$  be the list assignment defined by  $L'(x_1) = \{c_1\}$ ,  $L'(x_2) = \{c_1, c_2\}$ ,  $L'(y_i) = L(y_i) \setminus \{c_1, c_2\}$  for  $i = 1, 2$  and  $L'(v) = L(v)$  for every  $v \in V(G) \setminus \{x_1, x_2, y_1, y_2\}$ . Then  $L'$  is a suitable list assignment of  $G \setminus C$ . Hence  $G \setminus C$  admits a proper  $L'$ -colouring, which is an  $L$ -colouring of  $G$  by the definition of  $L'$ .  $\square$

**PROBLEM 6.10.** *Is every graph with crossing number 2 five-choosable?*

**6.4. Graphs with small clique number or large girth.** The celebrated Grötzsch Theorem [9] asserts that triangle-free (i.e. with clique number at most 2) planar graphs are 3-colourable. (See also [19] for a short and elegant proof.) Together with Theorem 1.4, this suggests that the upper bounds on the chromatic number of graphs which are almost planar (they have few crossings or pairwise nondependent crossings or contain few edges whose deletion makes them planar) may be lowered when considering graphs with small clique number. We now prove a result analogous to Theorem 1.5 for  $K_4$ -free graphs.

**THEOREM 6.11.** *If a  $K_4$ -free graph has a drawing in the plane in which no two crossings are dependent, then it is 4-colourable.*

*Proof.* Consider a drawing of a  $K_4$ -free graph  $G$  in which no two crossings are dependent. Let  $C_i = \{u_i v_i, x_i y_i\}$ ,  $i \in I$ , be the crossings. Since  $G$  is  $K_4$ -free, without loss of generality, we may assume that  $u_i x_i$  is not an edge for all  $i \in I$ . Let  $G'$  be the graph obtained from  $G$  by identifying  $u_i$  with  $x_i$  for every  $i \in I$  into a vertex  $z_i$ . The graph  $G'$  is planar. Thus, by the Four Colour Theorem,  $G'$  admits a proper 4-colouring  $c'$ . Let us define  $c$  by  $c(x_i) = c(x_i) = c'(z_i)$  for every  $i \in I$  and  $c(v) = c'(v)$  for every vertex  $v \in V(G) \cap V(G')$ . Since, for every  $i \in I$ ,  $x_i$  and  $u_i$  are not adjacent,  $c$  is a proper 4-colouring of  $G$ .  $\square$

Note that Theorem 6.11 is tight because there exist  $K_4$ -free planar graphs which are not 3-colourable. But can it be improved for triangle-free graphs or is there a triangle-free graph which has a drawing in the plane in which no two crossings are dependent and which is not 3-colourable?

For triangle-free graphs, one can show an analogue to Theorem 1.6.

**THEOREM 6.12.** *If a triangle-free graph contains a set of (at most) four edges whose deletion results in a planar graph, then it is 4-choosable.*

*Proof.* By induction on the number  $n$  of vertices of the triangle-free graph  $G$ , the result holding trivially when  $n \leq 4$ . A triangle-free planar graph on  $n$  vertices has at most  $2n - 4$  edges. Hence  $G$  has at most  $2n$  edges. Thus either  $G$  has a vertex  $v$  of degree at most three or it is 4-regular.

In the first case, by the induction hypothesis  $\text{ch}(G - v) = 4$ . Let  $L$  be a 4-list-assignment of  $V(G)$ . The graph  $G - v$  admits an  $L$ -colouring  $c$  that can be extended to  $G$  by assigning to  $v$  a colour in its list not assigned to any of its neighbours. So  $G$  is 4-choosable.

In the second case, since  $G$  is triangle-free it contains no  $K_5$  and thus by Brooks' Theorem for list-colouring,  $\text{ch}(G) \leq 4$ .  $\square$

For  $\mathcal{C}_3$  and  $K_4$  and more generally, for any graph or any family of graphs  $\mathcal{F}$ , one can ask the following questions.

PROBLEM 6.13. What is the smallest integer  $f_{\mathcal{F}}(k)$  (resp.  $g_{\mathcal{F}}(k)$ ) such that every  $\mathcal{F}$ -free graph  $G$  and crossing number at most  $k$  is  $f_{\mathcal{F}}(k)$ -colourable (resp.  $g_{\mathcal{F}}(k)$ -choosable)?

In particular, for  $\mathcal{C}_g = \{C_i | i = 3, \dots, g - 1\}$  the family of cycles of length less than  $g$ , the  $\mathcal{C}_g$ -free graphs are graphs with girth at least  $g$ . Set  $f_g(k) = f_{\mathcal{C}_g}(k)$ . Trivially,  $f_g(k) \leq f_h(k)$  if  $g \geq h$ . In particular, for any  $g \geq 3$ ,  $f_g(k) \leq O(k^{1/4})$  since  $f_3(k) = f(k) = O(k^{1/4})$ . Erdős [6] showed that there are graphs with arbitrarily large girth and chromatic number. Hence for any fixed  $g$ ,  $f_g(k)$  tends to infinity when  $k$  tends to infinity. The Grötzsch graph is triangle-free, has crossing number at most 5 and chromatic number 4, so  $f_3(5) \geq 4$ . Thomas and Walls [15] proved that every graph of girth at least five which admits an embedding in the Klein bottle is 3-colourable. Since every graph with crossing number at most 2 is embeddable in the Klein bottle, it follows that every graph of girth at least 5 and crossing number at most 2 is 3-colourable.

Jensen and Royle [10] showed a  $K_4$ -free graph with crossing number at most 6 and chromatic number 5, so  $f_{K_4}(6) \geq 5$ .

One can prove an analogue to Theorem 1.5 for graphs of large girth.

PROPOSITION 6.14.

*Let  $G$  be a graph having a drawing in the plane in which no two crossings are dependent.*

*(i) If  $G$  has girth at least 5, then  $G$  is 4-choosable.*

*(ii) If  $G$  has girth at least 10, then  $G$  is 3-choosable.*

*Proof.* Let us prove that  $G$  has a vertex of degree at most 3 (resp. at most 2) if  $G$  has girth at least 5 (resp. 10). Then an easy induction would give the result.

Let  $n$  be the number of vertices of  $G$ . Since no two crossings are dependent, then  $G$  has at most  $n/4$  crossings. Hence there is a set  $F$  of at most  $n/4$  edges such that  $G \setminus F$  is planar. Moreover,  $G \setminus F$  has girth at least 5 (resp. 10), so  $G \setminus F$  has fewer than  $\frac{10}{6}n$  (resp.  $\frac{3}{4}n$ ) edges. Hence  $G$  has fewer than  $\frac{23}{12}n$  (resp.  $n$ ) edges. Hence  $G$  has a vertex of degree at most 3 (resp. 2).  $\square$

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